A probabilistic protocol for the assessment of transition and control

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Transition to turbulence dramatically alters the properties of fluid flows. In most canonical shear flows, the laminar flow is linearly stable and a finite-amplitude perturbation is necessary to trigger transition. Controlling transition to turbulence is achieved via the broadening or narrowing of the basin of attraction of the laminar flow. In this Letter, a novel methodology to assess control strategies is introduced that relies on the statistical sampling of the phase space neighborhood of the laminar flow in order to assess the transition probability of perturbations as a function of their energy. This approach is applied to plane Couette flow in the presence of an established control strategy and reveals counter-intuitive results: transition is significantly suppressed whereas usual key indicators of the nonlinear stability of the flow, such as the edge state energy, suggest the opposite. The methodology presented here is easily implementable for any flow configuration and will be of interest for a wide range of applications.

Controlling transition to turbulence is vitally important in a variety of applications ranging from pipeline flows to mixing devices. In the former, turbulence undesirably enhances transport losses, whilst in the latter, it improves mixing efficiency. A major ongoing effort to control transition aims to manipulate the robustness of the laminar flow to perturbations via passive modifications of channel surfaces and linear feedback control techniques [1,7]. The resulting strategies are largely based on the knowledge that, in subcritical shear flows, transition to long-lived turbulence occurs when both the Reynolds number, quantifying the ratio of the inertia of the flow to the viscous forces, exceeds a critical value $Re_c$ [8,9], and a sufficiently energetic disturbance is applied [10,11]. A strategy designed to prevent transition can be deemed successful when it increases the volume of the basin of attraction of the laminar flow. This volume may be related to the edge of chaos, the manifold separating initial conditions that decay from those that transition [12,13]. The edge of chaos comprises local attractors, the edge states, and their stable manifolds. The importance of edge states has indeed been highlighted for boundary layer and pipe flows, where their neighborhood is often visited before transition [15,16], and for plane Couette flow, where they have been related to optimal energy growth disturbances [17]. Edge states, together with minimal seeds, i.e., the minimal energy perturbations that trigger turbulence [18,19], can therefore be thought of as important indicators of the robustness of the laminar flow to finite amplitude perturbations.

A tempting way to design control strategies for transition is via the maximization of the energy of the minimal seed or of the edge state [17]. In this Letter, we show that such quantities are not sufficient to provide a reliable conclusion about the efficiency of control strategies. To address such shortcomings, we introduce a novel methodology: the statistical sampling of the phase space to determine the probability that perturbations laminarize as a function of their energy. We apply this approach in plane Couette flow where control is imposed via transverse wall oscillations [2,4,7].

We consider plane Couette flow (pCf), i.e., the flow confined between two infinite plates separated by a gap $2h$ and moving in opposite directions with constant velocity $U$. The domain is periodic in the streamwise (period $\Gamma_x = 4\pi h$) and in the spanwise (period $\Gamma_z = 32\pi h/15$) directions, and no-slip boundary conditions are applied at the walls [19]. The nondimensionalized Navier–Stokes equation and the incompressibility condition are given by:

$$\partial_t u + v e_x + y \partial_x u + (u \cdot \nabla)u = -\nabla p + \frac{1}{Re} \nabla^2 u, \quad (1)$$

$$\nabla \cdot u = 0, \quad (2)$$

where $Re = U h / \nu$ is the Reynolds number, $\nu$ is the kinematic viscosity, $p$ is the pressure, $e_x$ is the unit vector in the streamwise direction and $u$ satisfies homogeneous boundary conditions in $y$. To obtain these equations, we have decomposed the nondimensional velocity $U = U_{lam} + u$, where the laminar flow $U_{lam} = ye_x$ and $u$ is the incompressible perturbation. The no-slip boundary conditions read: $u(y = \pm 1) = 0$.

We use Channelflow [20] to solve for the flow at $Re = 500$, a value larger than that necessary to observe sustained turbulence—the critical value is $Re_c = 325 \pm 10$ [9,21]. For this value of the Reynolds number, we calculate the time-averaged kinetic energy of turbulent flow realizations: $E_{turb} \approx 6 \cdot 10^{-2}$, where the instantaneous kinetic energy $E$ for the state $u$ is defined as

$$E = \frac{1}{2} \langle u, u \rangle = \frac{1}{2} ||u||^2 = \frac{1}{2 V} \int \Omega u \cdot u \partial \Omega, \quad (3)$$

where $V = 2 \Gamma_x \Gamma_z = 256 \pi^2/15$ is the volume of the domain $\Omega$. We time-integrate a number of initial conditions and assess their behavior in the following way. If the kinetic energy of the resulting flow exceeds $E_{turb}$ at any point and the flow does not laminarize in $t_{turb} = 400$ time units, the flow is said to have transitioned to turbulence.
Conversely, if the energy decays below $E_{lam} = E_{turb}/100$, the flow is said to have laminarized. The role of the non-zero waiting time $t_{turb}$ is to ensure that an event where the flow exceeds $E_{turb}$ and decays immediately after is not counted as transitional.

We compute $P_{lam}(E^{(j)})$, the laminarization probability of a random initial perturbation (RP) of energy $E^{(j)}$. To do so, we consider 40 equispaced energy levels $E^{(j)}, j = 1, \ldots, 40$, between 0 and 0.04. The edge state kinetic energy $E_{edge} \approx 1.82 \times 10^{-2}$ is computed using edge tracking [12], as shown in figure 1(a). For each energy level, we generate 200 RP, $u^{(j,k)}, k = 1, \ldots, 200$, which we time-integrate until one of the aforementioned energy thresholds is crossed to determine the type of flow. We approximate $P_{lam}(E^{(j)})$ by the fraction of laminarizing RP of energy $E^{(j)}$.

We express the RP as a linear combination of the laminar flow field $U_{lam}$ and an incompressible orthogonal component $u^{(j,k)}$, i.e. $u^{(j,k)} = Au^{(j,k)} + BU_{lam}$, where $A$, $B$ and $u^{(j,k)}$ are generated randomly, $\langle u^{(j,k)}, U_{lam} \rangle = 0$ and $||u^{(j,k)}|| = 1$. To ensure that $u^{(j,k)}$ has energy $E^{(j)}$, we further impose $||Au^{(j,k)} + BU_{lam}||^2 = 2E^{(j)}$. Turbulence in pCf is associated with shear concentration at the wall, yielding lower kinetic energies for the turbulent states than for the laminar flow. We thus find it intuitively useful to distinguish RP with weakened bulk shear ($B < 0$) from those that display stronger bulk shear ($B \geq 0$), the sets of which are hereafter called RP− and RP+, respectively. The former can be thought of as being mostly located in phase space between the laminar (highly energetic) and the turbulent (energetically lower) flows, while the latter are located farther away from turbulence. We create such perturbations in three steps. First, we generate the random orthogonal component $u^{(j,k)}$ by drawing its spectral coefficients from the uniform distribution so that the homogeneous boundary conditions are satisfied and $||u^{(j,k)}|| = 1$. Next, we draw $B$ from the uniform distribution between $-2E^{(j)}/||U_{lam}||$ and $2E^{(j)}/||U_{lam}||$. Lastly, we compute $A = \pm \sqrt{2E^{(j)} - B^2||U_{lam}||^2}$. To ensure fair sampling, we use an equal number of positive and negative $A$ and $B$ for each energy level.

The main results are reported in figure 2. For sufficiently small initial disturbance energies, the probability of laminarization tends to 1, owing to the fact that the laminar flow is linearly stable. The first transitioning RP was found at the third energy level $E^{(3)} \approx 4 \times 10^{-3}$, implying that the minimal seed has energy $E_{min} \leq 4 \times 10^{-3}$. The laminarization probability decreases nearly monotonically with the RP energy to saturate at $P_{lam} \approx 0.08$. Most of the laminarization probability decay occurs at
small amplitude, i.e., for \( E < E_{\text{edge}} \) \( (P_{\text{lam}}(E_{\text{edge}}) \approx 0.12) \). For the most part, RP— is responsible for the non-vanishing laminarization probability as the energy increases. The fact that \( P_{\text{lam}} \) does not tend to zero even for large energies is a consequence of the structure of the edge: it is wrapped around the turbulent saddle such that laminarizing and transitioning RP regions are locally interleaved [14]. To understand further which RP laminarizing and transitioning RP regions are locally in-edge: it is wrapped around the turbulent saddle such that for large energies is a consequence of the structure of the edge. To understand further which RP laminarizing and transitioning RP regions are locally in-edge: it is wrapped around the turbulent saddle such that for large energies is a consequence of the structure of the edge.

The probability distribution for the uncontrolled case can be approximated by the cumulative distribution function for the Gamma distribution centered at 0.5 and saturated at 0.0805, i.e. \( p(E) = 1 - (1 - a)\gamma(\alpha, \beta E) \), where \( \alpha = 2.05 \), \( \beta = 412 \), \( a = 0.0805 \) and \( \gamma(\alpha, \beta E) \) is the lower incomplete gamma function. The various coefficients have been determined via least-square fitting and the resulting function is shown in figure 2 by the solid line. One can assess control strategies by simple quantitative comparison with this distribution. If the action of a control strategy leads to an increase of the laminarization probability, it is successful and its efficiency can be quantified by the probability gain, which not only approximates the size of the increase of the basin of attraction of the laminar flow, but also reveals important information about the sensitivity of the laminar flow to perturbations of various amplitude.

To demonstrate this, we impose in-phase spanwise wall oscillations, a strategy known to reduce the turbulent drag [3] and increase the energy of the minimal seed [7]. Under these oscillations, the modified boundary conditions read: \( \mathbf{U}(y = \pm 1) = [\pm 1, 0, A \sin(\omega t + \phi)] \), where \( A \), \( \omega \) and \( \phi \in [0; 2\pi] \) are the amplitude, the frequency and the phase of the oscillations. The laminar flow becomes oscillatory and equation (1) is changed into equation (2.5) from [7]. We use the parameter values close to the optimal ones found by Rabin et al.: \( A = 0.3 \) and \( \omega = 1/16 \) [7]. For these parameter values and domain size, we find that the edge state is chaotic, as shown in figure 1(b), with average kinetic energy \( E_{\text{edge}} = 1.15 \cdot 10^{-2} \), approximately 37% less than for the uncontrolled edge state.

We generate the RP in the same way as for the uncontrolled case, except that we also impose a random phase \( \phi \) drawn from a uniform distribution between 0 and \( 2\pi \).

The laminarization probability for the controlled case is shown in figure 3. The resulting probability distribution decreases nearly monotonically to saturate around \( P_{\text{lam}} = 0.3 \) for large energy RP. This behavior is qualitatively similar to that observed in the absence of control but fundamental quantitative differences can be reported. Firstly, the probability distribution plateaus at lower RP energy than in the uncontrolled case. This results in a larger asymptotic value of \( P_{\text{lam}} \), more than double its value in the absence of wall oscillation. Secondly, a non-negligible fraction of large energy initial perturbations of RP+ are now found to laminarize. Oscillating the walls in the spanwise direction did not statistically af-
fect the behavior of small energy perturbations from the laminar flow but rather increased the probability that large energy ones laminarize. To shed more light on this phenomenon, we show, in figure 5, a similar representation of the initial conditions as in figure 3 but for the controlled case. We recover the small $A$ region where laminarization was found in the uncontrolled case, however, we also found laminarizing RP at larger amplitude, scattered around the region where all RP transitioned in the absence of control. These newly controlled RP decay via overshooting, i.e., their energy first significantly exceeds $E_{turb}$ before decaying nearly monotonically. The initial phase of the RP did not seem to play any role in determining whether or not the flow will laminarize.

The laminarization probability in the controlled case can be approximated by the fitting function $p_{osc}(E)$, which shares the same structure as $p(E)$ but with $\alpha = 3.75$, $\beta = 899$, $a = 0.286$. The probability gain can thus be computed as $p_{osc}(E) - p(E)$ averaged over the range of the considered energies, which is equal to 0.165, informing that, on average, 16.5\% of the RP have been controlled. When a single value is not satisfactory to assess the control efficiency, more detailed information can be obtained via inspecting the differences between the laminarization probabilities at each energy level.

While the fate of perturbations in the uncontrolled flow is sensitive to the strength of the initial spanwise shear, this is no longer the case in the presence of control via spanwise wall oscillation. Furthermore, the energy of the edge state decreases under the oscillating wall effect. Without any further knowledge, this could be interpreted as the failure of the control strategy to postpone transition but the more exhaustive analysis of phase space provided here shows that this strategy is indeed effective. The use of the minimal seed does not seem to provide a better basis for control assessment: the shape of the basin of attraction of the laminar flow is such that, in this study, most perturbations generated with 4 times the energy of the minimal seed laminarized, making it difficult to extrapolate any reliable information. These observations further strengthen our approach to consider a more exhaustive method to assess control.

In this Letter, we have introduced a new way to analyze the robustness of the laminar flow to perturbations and assess control strategies. We proceeded by sampling phase space in the neighborhood of the laminar flow and evaluating the probability that the sampled initial conditions laminarize as a function of their initial energy. Our results for plane Couette flow in a small domain indicate that the laminarization probability decreases with the kinetic energy of the initial perturbation from the laminar flow and that the resulting probability distribution can easily be recomputed and compared in the presence of control. We tested this methodology under control via spanwise wall oscillations and observed a neat increase in the laminarization probability, a result that would not have been anticipated by the consideration of scalar criteria such as the energy of the edge state. We introduced this methodology for plane Couette flow but anticipate that it will prove useful to control in other flow configurations [22, 25].