Transition to doubly diffusive chaos

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Abstract

Doubly diffusive convection is studied in a three-dimensional enclosure of square horizontal cross-section. Previous studies have focused on the emergence of steady states and revealed that the primary instability from the conduction state gives rise to subcritical branches of spatially localized states that undergo snaking. These states are further destabilized by the presence of the twist instability, resulting in the absence of stable steady states beyond the primary bifurcation. This paper investigates the temporal dynamics in the vicinity of the primary bifurcation. When the conduction state is unstable, a sequence of instabilities occurs that gives rise to chaotic dynamics. This chaos is produced at a crisis bifurcation located close to the primary bifurcation and is characterized via its critical exponent. This phenomenon necessitates very few requirements to be observed, which is exemplified by the construction of a low-dimensional model, and is thus believed to be observable in many other systems.
INTRODUCTION

Doubly diffusive convection, the convection of a fluid driven by two density gradients, is a fascinating configuration that displays a wide range of behavior, some of which is still poorly understood. When it is used to model ocean dynamics, where the flow is driven by temperature and salt variations, it is called thermohaline convection and can lead to the formation of salt fingers and contribute to mixing currents [1–3]. It has also been used to characterize flows in magma chambers and at the Earth’s core-mantle boundary [4] and to model astrophysical transport [5].

Doubly diffusive convection is also a paradigm for pattern formation in fluids. Horizontal fluid layers placed within negative vertical gradients of temperature and salinity, unlike Rayleigh–Bénard convection, yield primary bifurcations to standing and traveling waves, some of which may be subcritical [6]. Under certain circumstances, the first bifurcation to steady states can also be found to be subcritical and, in the case of spatially extended domains, one can find bifurcations to spatially localized states which take the form of convection rolls in a well-bounded region in space, surrounded by motionless fluid [7]. This configuration has been investigated in more details under the additional Soret effect, in particular inspired by experiments by Kolodner [8–10]. A variety of time-dependent states were found numerically together with the discovery of steady localized binary fluid convection [11, 12]. The focus was then turned to the study of these localized states, called convectons. Their formation and bifurcation diagram was characterised by Mercader et al. [13, 14]. Asymmetrical perturbations, such as in the boundary condition [15] or the inclination of the fluid layer [16], were found to alter convectons by making them travel in space. Other time-dependent convectons have been observed in the presence of fixed temperature boundary conditions [17]. Watanabe et al. [18] revealed that convectons can be dynamically connected, showing the interplay between their stable and unstable manifold in a two dimensions configuration.

A number of routes via which a dynamical system can reach chaos have been documented, particularly in low-dimensional systems [19]. For example, the Ruelle–Takens–(Newhouse) scenario [20, 21] consists in a succession of at least three Hopf bifurcations from a steady state, leading to the appearance of broad band noise, a characteristic feature indicating the formation of a strange attractor. Another simple route to chaos is the period-doubling
cascade, occurring when a periodic orbit undergoes a series of pitchfork bifurcations taking place at well-defined intervals [22, 23] and resulting in the increase of the period of the orbit until it tends to infinity. Lastly, the Pomeau–Manneville scenario is associated with a saddle-node bifurcation, which, when crossed to the side where real-valued solutions no longer exist, leads to intermittency [24, 25]. The Lorenz system, which was derived in the context of Boussinesq convection [26] but can also be derived from other systems (e.g. the segmented disc dynamo [27], magnetoconvection [28] and porous medium convection [29]), gives an example of the variety of routes to chaos that a single system can exhibit. A number of period-doubling cascades can be observed for the most studied parameter values \((\sigma = 10, b = 8/3)\) and at large enough \(r\) (see chapter 4 of [30]), creating and destroying the strange attractor but it is a different route that generates chaos at low \(r\) involving a homoclinic bifurcation followed by a boundary crisis [31, 32]. Although convection problems sometimes reduce to scenarios similar to that of simple systems, such as the aforementioned Lorenz system, they are not limited to it. Meca et al. observed a blue-sky catastrophe in the doubly diffusive convection of a horizontal fluid layer with Soret effect [33] and Gao et al.’s study of the natural convection in a differentially heated vertical fluid layer revealed different routes to chaotic dynamics: the progressive addition of temporal frequencies in two dimensions and a period-doubling cascade in three dimensions [34].

We are here interested in doubly diffusive convection in a vertical closed container. The study of this system in an extended two-dimensional domain led to the discovery of steady spatially localized states organized along two subcritical homoclinic snaking branches. Some of these solutions are stable in two dimensions and perturbing them to the right the snaking region revealed a depinning instability by which the flow visits a series of localized states of increasing size until a stable domain filling state is reached [35]. Small aspect-ratio three-dimensional domains revealed transition to chaos via a sequence of period-doubling bifurcations leading to a symmetric chaotic attractor that then undergoes an internal crisis to become asymmetric [36]. Larger three-dimensional domains revealed richer pattern formation than the two-dimensional domains with new families of localized states and the presence of chaos immediately above the primary onset [37, 38].

The aim of this paper is to characterize the transition to chaos in the large three-dimensional domain. The problem is formalized mathematically in Section 2, followed, in Section 3, by the description of the successive instabilities leading from conduction to
chaos. Section 4 is devoted to the characterization of the characteristic times related to the transition. The paper terminates with a low-dimensional model of the scenario and with a discussion.

**PROBLEM SET-UP**

We investigate the instability of a binary fluid mixture placed in a closed three-dimensional container. The container has a square cross-section and is extended in the vertical, $x$, direction with aspect ratio $L = 19.8536$. The two opposite vertical walls at $z = 0$ and $z = l$ are maintained at fixed temperatures and salinities while the other walls are modelled using no flux boundary conditions for both temperature and salt concentration. The fluid obeys the Navier–Stokes equation under the Boussinesq approximation, the incompressibility condition, the heat equation as well as an advection/diffusion equation for the salinity. Upon nondimensionalisation, these equations read:

\[
Pr^{-1} [\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + Ra(T + NC) e_x + \nabla^2 \mathbf{u}, \tag{1}
\]

\[
\nabla \cdot \mathbf{u} = 0, \tag{2}
\]

\[
\partial_t T + (\mathbf{u} \cdot \nabla) T = \nabla^2 T, \tag{3}
\]

\[
\partial_t C + (\mathbf{u} \cdot \nabla) C = \tau \nabla^2 C, \tag{4}
\]

where $t$ is the time, $\mathbf{u}$ is the velocity field, $p$ is the pressure, $T$ is the temperature difference with the temperature at the $z = 0$ wall and $C$ is the salinity difference with the salinity at the $z = 0$ wall. In writing these equations, the Prandtl number $Pr$, the Rayleigh number $Ra$, the inverse Lewis number $\tau$ and the buoyancy ratio $N$ have been defined as:

\[
Pr = \frac{\nu}{\kappa}, \quad Ra = \frac{g|\rho_T|\Delta T^3}{\rho_0 \nu \kappa}, \quad \tau = \frac{D}{\kappa}, \quad N = \frac{\rho_C \Delta C}{\rho_T \Delta T}, \tag{5}
\]

where $\nu$ is the kinematic viscosity of the fluid, $\kappa$ is the thermal diffusivity of the fluid, $g$ is the gravitational acceleration, $\rho_0$ is the fluid density at the reference temperature, $\rho_T/\rho_0$ is the thermal expansion coefficient in the Boussinesq approximation evaluated at the reference temperature, $\rho_C/\rho_0$ is the solutal expansion coefficient in the Boussinesq approximation evaluated at the reference salinity, $\Delta T$ (resp. $\Delta C$) is the temperature (resp. salinity) difference between the $z = 0$ and $z = l$ walls and $D$ is the molecular diffusivity of the fluid. We consider the case $N = -1$ so that the solutal and thermal contributions to the
buoyancy compete with equal strength. This special case allows for the equations to admit the conductive solution \( u = (u, v, w) = 0, T = C = z \) and to be equivariant with respect to the dihedral group \( D_2 \) comprised of the identity \( I \) and of the reflections:

\[
S_y : [u, v, w, \Theta, \Sigma](x, y, z) \rightarrow [u, -v, w, \Theta, \Sigma](x, 1 - y, z), \tag{6}
\]

\[
S_\Delta : [u, v, w, \Theta, \Sigma](x, y, z) \rightarrow -[u, -v, w, \Theta, \Sigma](-x, y, 1 - z), \tag{7}
\]

where \((\Theta, \Sigma) = (T - z, C - z)\) are the so-called convective variables. The composition of both the above reflections is hereafter called \( S_c \).

When posed in two dimensions, this system [35] only possesses the \( S_\Delta \) symmetry (thus yielding a \( Z_2 \)-equivariant system, or an \( O(2) \)-equivariant in the case of spatially periodic boundary conditions), the solutions being considered invariant in the \( y \) direction. The instability from the conduction state leads to steady domain-filling convection via a hysteresis. The presence of the \( S_y \) symmetry in the three-dimensional system is responsible for a major enrichment of the bifurcation diagram and of the observed dynamics. Indeed, an \( S_y \) breaking instability of the convection, called twist instability [38], leads to the complete destabilization of the hysteretic states, which leads to the absence of stable steady state for parameter values beyond critical. In the following section, we shed light on how the presence of the \( S_y \) symmetry impacts this instability and leads to the establishment of chaotic dynamics.

**INSTABILITY FROM THE CONDUCTIVE STATE**

The conductive state \((u = v = w = 0)\) is linearly stable until \( Ra \approx 850.78 \), where convection develops through an instability resulting from two close bifurcations. Both bifurcations are created by eigenvectors leading to the formation of an array of counter-rotating rolls growing from the center [37]. These two eigenmodes, shown in figure 1(a,b), are different in a subtle way. On the one hand, the eigenmode in figure 1(b) is fully symmetric and creates a transcritical bifurcation at \( Ra \approx 850.86 \) that is responsible for the creation of a wide variety of localized states [37, 38]. On the other hand, the eigenmode in figure 1(a) has a similar structure but with a phase shift of a quarter of the wavelength. As a result, it breaks both the \( S_\Delta \) and the \( S_c \) symmetry, and only preserves \( S_y \). This eigenmode is responsible for the creation of a (subcritical) pitchfork bifurcation at \( Ra \approx 850.78 \) [38]. These two eigenmodes have similar growth rate (e.g. \( \lambda \approx 0.20 \) at \( Ra = 930 \), where \( \lambda \) is the temporal growth rate).
FIG. 1. Most unstable eigenmodes of the conductive state at \( Ra = 930 \) shown using two opposite contours of \( x \)-velocity. Eigenmode (a) is the first to become destabilizing at \( Ra \approx 850.78 \), eigenmode (b) is the one producing the subcritical patterns shown in [37, 38], becoming destabilizing at \( Ra \approx 850.86 \). Eigenmodes (c) and (d) are two-pulse modes that are marginal at \( Ra \approx 859 \), and eigenmodes (e) and (f) are three-pulse modes marginal at \( Ra \approx 872 \). In all figures, the gravity bearing direction \( x \) is represented horizontally and the direction bearing the temperature and salinity gradients, \( z \), is represented vertically.

and remain the most unstable eigenmodes of the conductive state at all the values of the Rayleigh number tested. Other modes become destabilizing at higher values of the Rayleigh number: bifurcations to two-pulse states are found at \( Ra \approx 859 \) and to three-pulse states at \( Ra \approx 872 \). The eigenmodes related to these bifurcations are represented for \( Ra = 930 \) in figure 1. At this value of the Rayleigh number, the two-pulse eigenmodes have growth rate \( \lambda \approx 0.18 \) while that of the three-pulse eigenmodes have growth rate \( \lambda \approx 0.15 \). Crucially, all these eigenmodes preserve the symmetry \( S_y \) and trigger the formation of rolls with rotation axis \( e_y \), i.e., with dominant velocities in \( x \) and \( z \). Figure 2 shows the growth rate of the most unstable eigenmodes for \( 845 \leq Ra \leq 950 \). The instability hierarchy is preserved, in the parameter region investigated: the one-pulse perturbations is always the leading linear instability.

As the instability grows away from the linear regime, the rolls displaying downflow close
FIG. 2. Most unstable eigenvalues in the range $845 \leq Ra \leq 950$ represented using the exponential of the real part of the eigenvalue $e^{\lambda_r} = e^{\text{Re}(\lambda)}$. The thick line at $e^{\lambda_r} = 1$ represents marginality. The two eigenvalues representing the one-pulse eigenmodes are labelled 1p, those representing the two-pulse eigenmodes are labelled 2p and those representing the three-pulse eigenmodes are labelled 3p. In most cases, eigenvalues of the same nature are undistinguishable on the figure.

to the hot and saline wall are suppressed in favor of the other rolls, giving rise to convective structures comprised of co-rotating rolls [39]. Owing to the presence of no-slip walls at $x = 0$ and $x = L$, the structure grows from the center of the domain and propagates outwards by the successive nucleation of convection rolls to fill the entire domain. The growth of the instability is exemplified at $Ra = 880$ in figure 3. A preliminary simulation was initialized by the conductive state so that the instability developed from numerical noise. When the instability grew to a domain-integrated kinetic energy of about $10^{-4}$, the solution was stored and two new simulations were produced: one starting from that state and shown in panels (a,c) and the other one (panels (b,d)) with a sign flip to produce an initial condition on the opposite side of the stable manifold of the conduction state. The results are shown through two dynamics indicators:

$$A_{\text{conv}}(x, t) = \sqrt{\int_0^1 \int_0^1 u^2 dy dz}, \quad (8)$$
FIG. 3. Spatiotemporal growth of the convective pattern triggered by numerical noise at $Ra = 880$ and represented by an indicator of the convection amplitude $A_{\text{conv}}$ (see equation (8)) in panels (a,b) and an indicator of the tilt amplitude $A_{\text{tilt}}$ (see equation (9)) in panels (c,d) as a function of the time $t$ and vertical coordinate $x$. The simulation in (b,d) was initialized by a small amplitude solution from the simulation in panel (a,c) whose sign was flipped to produce an initial condition on the opposite side of the stable manifold of the conductive state. Time was reset at the start of each simulation.

$$A_{\text{tilt}}(x, t) = \sqrt{\int_0^1 \int_0^1 v^2 dy \, dz}, \quad (9)$$

where $u = u(x, y, z, t)$ represents the velocity in the vertical ($x$) direction and $v = v(x, y, z, t)$ is the velocity component in the $y$ direction. Indicator $A_{\text{conv}}$ is designed to quantify the amplitude of convection owing to the fact that $u$ typically represents the largest magnitude velocity component of the flow by nearly an order of magnitude. The rotation axis of the rolls emerging from the instability is $e_y$ and, thus, their velocity has major components in the $e_x$ and $e_z$ directions with negligible velocity in the $e_y$ direction. However, if a convection roll tilts around the $e_x$ axis (i.e., its rotation axis obtains a non-zero component on $e_z$), a non-negligible velocity component in the $e_y$ direction is created. Indicator $A_{\text{tilt}}$ is designed to
FIG. 4. Snapshot from the simulation in figure 3(b,d) taken at $t = 72.5$. The state is represented using two opposite values of the velocity: $u = \pm 1$.

capture the tilt of the rolls: in the absence of tilt, $A_{\text{tilt}} \ll 1$. The first simulation, presented in figure 3(a,c), produces a structure consisting of 4 co-rotating rolls at $t = 50$ before the growth a fifth one at $t \approx 55$ at the top of the structure and a sixth one at the bottom of the structure at $t \approx 67$, at which point convection fills the domain. The other simulation, in figure 3(b,d), produces an array of 5 co-rotating rolls at $t = 50$ whose dynamics slows down until the nucleation of two rolls at $t \approx 67$ and $t \approx 70$, leading to a domain-filling 7-roll state before the emergence of temporal complexity. Owing to the clear separation between the growth rates of the 1-pulse modes and the others, simulations initialized by a small perturbation around the conductive state at Rayleigh numbers close enough to the instability threshold invariably lead to one or the other above scenarios. The resulting domain-filling is $S_y$-symmetric and is reminiscent from spatially periodic convection states [35]. The number of rolls comprising the domain-filling state depends on the projection of the initial condition onto the unstable manifold of the conductive flow. One side of the stable manifold leads to the presence of a roll in the center of the domain, and, thus, of an odd roll count, while the other side leads to a gap in the center of the domain and an even roll count.

A secondary instability further arises on the domain-filling state that directly leads to temporal complexity. This instability affects each roll individually by tilting them around the vertical axis so that their axis develops a non-zero component on $e_z$ in addition to its dominant component on $e_y$. This effect of this instability is exemplified in figure 4 by a snapshot of a simulation ran at $Ra = 880$ displaying a domain-filling state consisting in 7 convection rolls. Most these rolls produce a flow rotating around $e_y$, except for the central roll which is tilted and whose axis is no longer parallel to $e_y$ but has now a non-negligible projection onto $e_z$. Owing to the tilt, the high-velocity regions of the central roll are no longer contiguous with the forcing walls. The roll can no longer be maintained and decays
FIG. 5. Growth of the instability in an artificially constrained domain designed to only allow the initial growth of one central roll. The same simulation as in figure 3(b,d) is used but with artificial damping away from the center of the domain. The flow is represented in the central part of the domain through $A_{\text{conv}}$ (a) quantifying the strength of convection and the flow rate in the $y$ direction $\Phi_y$ (b).

shortly after tilting, as shown in figure 3. The next roll to be affected by the instability is the second roll formed: the one directly located to the right of the central one. The beginning of this second tilting event is visible in figure 4. Figure 3 shows that all the initial rolls survive for a duration of the order of 20 time units before tilting, decaying and the emergence of chaotic dynamics supported by shorter-lived rolls.

To investigate roll decay, a small amplitude state from the simulation shown in figure 3(b,d) is used as initial condition and the domain constrained by adding artificial damping for $x < 7.4451$ and $x > 12.4085$, allowing for the growth of only one initial roll. The results are shown in figure 5 through the quantities $A_{\text{conv}}$ and $\Phi_y$, where:

$$\Phi_y(x, t) = \int_0^1 \int_0^1 vdydz. \quad (10)$$

Because of the constraint imposed by the added artificial damping, only one roll grows from the linear instability. The added constraint slows down the growth of the instability: it took
FIG. 6. Snapshots from the simulation in figure 5 taken at $t = 292$ (a) and $t = 293$ (b). The states are represented using two opposite values of the velocity: $u = \pm 1$ in (a) and $u = \pm 0.8$ in (b). The orientation of the figure is similar to figure 4.

260 units of time for the central roll to reach its full amplitude. The roll then survives for about 30 time units before starting to tilt and slowly decaying at $t \approx 288$. By $t \approx 292$, maximum tilt is achieved, with a positive $y$-flow rate at its top and a compensating negative one at its bottom, as shown in figure 5(b) and in figure 6(a). The lack of constraints due to the walls or any other adjacent roll allows the localized roll to expand further than it would normally, as can be seen by comparing the roll size in figure 6(a) with that in figure 4. The tilt accelerates the decay of the roll and leads to breakup from its center, forming two tilted rolls of smaller amplitude at $t \approx 293$ (see figure 6(b)) that continue to decay, albeit more slowly, until the state reaches a minimum energy at $t \approx 296$. The signature of these secondary rolls is also visible in the space-time plot in figure 5(b): as the red and blue patches indicating non-zero $y$-flow rate open up, two lower-intensity patches of the opposite colors appear in between. In addition to highlighting the mechanisms by which the rolls decay, this provides insight into the formation of subsequent rolls. The roll breaks up into two smaller structures carrying residual momentum and providing seeds for two rolls when the linear instability of the conductive state is triggered again. In the above simulation, the two rolls that are produced at $t \approx 302$ are very close to the artificial edge of the domain and all subsequent dynamics are hence irrelevant. In a full domain simulation, interaction with the walls and between rolls dominate and lead to a strong partitioning of space: as the rolls do not tilt at the same time, when a roll decays, there is generally not enough room for two new rolls to form and the two seeds of energy quickly merge into one.
CRISIS BIFURCATION

The sudden presence of a finite amplitude difference in the observed end-states around the primary bifurcation point is typical of subcritical systems and is often accompanied with a hysteresis, as it is the case in two-dimensional doubly diffusive convection [35]. The fact that the configuration considered here is three-dimensional introduces the symmetry $S_y$ and allows for a secondary instability to take place, thereby destabilizing the steady state that would otherwise be an attractor. To identify the region of existence of the chaotic behavior, an arbitrary instantaneous state from the chaotic dynamics found at $Ra = 900$ was time-integrated while progressively decreasing the value of the Rayleigh number. Figure 7 shows the result of such a simulation superimposed onto the bifurcation diagram showing the branches of steady states. The steady states bifurcate subcritically from the conductive state at $Ra \approx 850.86$ and take the form of two pairs of intertwined branches of convectons $C$ which produce snaking between $Ra \approx 700$ and $Ra \approx 810$ before going to larger Rayleigh numbers. They are subject to secondary bifurcations producing a family of secondary branches of twisted convectons $TC$, who undergo snaking between $Ra \approx 740$ and $Ra \approx 820$ before, also, extending to large Rayleigh numbers. In this simulation starting at $Ra = 900$, the Rayleigh number is decreased in time at rate $\xi = 1/500$. The simulation, which is initialized by a snapshot from a chaotic simulation, initially displays large-scale oscillations spanning two orders of magnitude in the total kinetic energy:

$$E = \frac{1}{2} \int_0^1 \int_0^1 \int_0^L (u^2 + v^2 + w^2) \, dx \, dy \, dz,$$

and reaching values up to those of the most energetic convectons. This behavior persists down to $Ra \approx 850$ where the flow approaches the conductive state more closely before its final decay at $Ra \approx 842$. Other simulations have been run to check the dependence of the critical Rayleigh number with $\xi$ and revealed that lowering $\xi$ led to decay at larger values of the Rayleigh number, until about $Ra = 840$, after which stochastic oscillations due to the nature of the flow dominate.

The sudden decay of the chaotic behavior observed in the simulations exemplified in figure 7 is typical of a crisis bifurcation. To characterize this bifurcation, we look for the time-scale over which chaos persists beyond the crisis, i.e., for lower values of the Rayleigh number. As the flow is highly dependent on the initial condition, we resort to considering a large number of initial conditions [40–42]. We generate these by running a chaotic simulation at $Ra = 900$. 
FIG. 7. Bifurcation diagram representing the total kinetic energy $E$ (see text) as a function of the Rayleigh number $Ra$. The diagram shows branches of steady states in color: (red) the convection branches $C$ correspond to the branches $L^\pm$ in [37, 38], (blue) the twisted convection branches $TC$ correspond to the branches $L_{1,2}^\pm$ in [38]. The black line shows the result of a numerical simulation initialized at $Ra = 900$ with an arbitrary instantaneous solution obtained along a chaotic trajectory at $Ra = 900$ and whose Rayleigh number was time-adjusted as follows: $Ra = 900 - \xi t$ with $\xi = 1/500$.

for $2 \cdot 10^4$ time units. The density plot shown in Figure 8 represents this simulation in a projection of phase space onto the $x$-kinetic energy $E_u = A_{\text{conv}}^2$ and the $y$-kinetic energy $E_v = A_{\text{tilt}}^2$. The chaotic flow remains in a relatively small region with fuzzy boundaries. We select a large number of snapshots from this simulation as initial conditions to characterize the crisis. To ensure fair sampling, we collect the snapshots corresponding to times $t = 100n$ with $n \in \mathbb{N}^+$, which proved to represent satisfactorily the attractor: the snapshots cover the whole area occupied by the attractor and their density is higher in areas that are more often visited, as shown in figure 8.

We ran simulations initialized with the snapshots identified in figure 8 for a number of subcritical Rayleigh numbers, ranging from $Ra = 750$, i.e., far away from the crisis, to $Ra = 845$, i.e., close to the crisis. The time at which the chaotic transient decayed was
FIG. 8. Phase portraits of the flow obtained during a simulation at $Ra = 900$ represented by the quantities $E_u = A^2_{con}$ and $E_v = A^2_{tilt}$ in (a) linear and (b) log scale. In blue: the darker the area, the more often it is visited. The red squares indicate the 200 selected snapshots.

recorded by identifying when the total kinetic energy fell below $E < 10^{-3}$ for the first time. The distribution of survival times at a given value of $Ra$ is statistical, as shown in figure 9(a). All our simulations decayed back to the conductive state in finite time and the time-scale increased as the value of the Rayleigh number approached that of the crisis. Much insight can be gained from basic dynamical considerations. First of all, the simulations comprise 3 stages: (i) an initial transient where the initial condition taken at $Ra = 900$ travels towards the chaotic saddle at the value of the Rayleigh number where the simulation is run, (ii) an chaotic transient where the flow is trapped in the saddle and (iii) decay. The stage of interest here is the second one: the first stage is an artifact of the choice of initial condition and the third stage is dictated by the linear stability of the conductive flow. Here, it helps to think of chaotic saddle as a topological ball with a hole. Once inside the ball, the trajectory bounces against its walls until it “finds” the hole, at which point the simulation enters its decay stage. This probabilistic depiction of chaotic saddles has led them to be called leaky attractor in the context of transition to turbulence [43, 44] and proves very useful in providing insight into the complex situation at hand. The crisis bifurcation corresponds to the opening of the hole onto the ball and the event corresponding to the trajectory exiting the chaotic saddle can be assimilated to a memoryless Poisson process. Given the above insight, we can
FIG. 9. (a) Time $t_{\text{decay}}$ at which the simulation first reached values of the total kinetic energy $E < 10^{-3}$ as a function of the initial condition used for (black) $Ra = 790$, (red) $Ra = 834$ and (blue) $Ra = 844$. (b) Survival probability associated with similar datasets than in (a) but for (black) $Ra = 834$, (red) $Ra = 838$ and (black) $Ra = 844$. The straight lines represent a fit to law (12) with $t_0 \approx -2.982$ and $\tau \approx 174.399$ for $Ra = 834$, $t_0 \approx -60.066$ and $\tau \approx 467.821$ for $Ra = 838$ and $t_0 \approx -58.300$ and $\tau \approx 2415.18$ for $Ra = 844$.

Infer that the probability of the trajectory exiting the leaky attractor follows an exponential distribution and, thus, that the probability of a simulation not decaying before time $t$ follows its complementary cumulative distribution function:

$$p(t) = e^{-\frac{t - t_0}{\tau}},$$

(12)

where $t_0$ corresponds to the initial transient duration and $\tau$ is the characteristic time associated with the process. Both $t_0$ and $\tau$ are functions of $Ra$. We represent in figure 9(b)
FIG. 10. Characteristic time of the chaotic transients $\tau$ as a function of the distance to criticality $Ra_X - Ra$, where $Ra_X = 850.78$ is chosen to be the primary bifurcation point. The black dotted line corresponds to the numerics while the straight red line corresponds to the fit: $\tau \sim (Ra_X - Ra)^{-3}$.

The survival probability $p$ for three sets of simulations taken at $Ra = 834$, $Ra = 838$ and $Ra = 844$ together with the least square fit to the corresponding law (12). The data obtained through the flow simulations are well approximated by the exponential distribution, the only visible departures from the law being observed in the tail of the distributions in figure 9(b). This confirms the relevance of the above topological interpretation to the flow dynamics. Further examination reveals that $\tau$ increases with $Ra$, leading to longer and longer simulations as we approach the crisis. The last complete set of simulations was obtained for $Ra = 845$. All the 200 initial conditions decayed to the conduction state, with the slowest ones staying in the vicinity of the chaotic saddle for more than $t = 10^5$ time units. The timescales exhibited by simulations for larger $Ra$ are such that it was computationally too expensive to pursue the numerical effort beyond $Ra = 845$.

The values obtained for $\tau$ as a function of the Rayleigh number are shown in figure 10. Since the flow decays down to the conductive state for subcritical values of $Ra$ and that this state loses stability at $Ra \approx 850.78$, it is logical to assume that the characteristic time associated with the chaotic transient diverges for $Ra \leq 850.78$. The results are therefore represented against the Rayleigh number offset by $Ra_X$, which is an approximate value for the Rayleigh number at the crisis bifurcation. In practice, the data obtained does not allow the accurate determination of $Ra_X$, so we set $Ra_X = 850.78$. The statistical transient times
FIG. 11. (a) Initial transient time $t_0$ and (b) its absolute value normalized by the characteristic time $\tau$ as a function of the Rayleigh number $Ra$.

$\tau$ are remarkably well fitted for $Ra_x - Ra < 50$, i.e., for $Ra > 800$ by:

$$\tau \propto (Ra_x - Ra)^{-\gamma},$$

(13)

where $\gamma \approx 3$. This exponent is larger than the ones typically observed for crisis bifurcations in fluids. For example, Zammert and Eckhardt found two crises in plane Poiseuille flow: an interior crisis associated with $\gamma = 0.8$ and a boundary crisis with $\gamma = 1.5$ [41], and Kreilos et al. observed a boundary crisis in plane Couette flow with $\gamma = 2.1$ [44]. Figure 11 shows the evolution of the initial transient time $t_0$ as a function of the Rayleigh number. The initial transient lasts approximately 10 time units at $Ra = 750$, the lowest Rayleigh number used, and grows monotonically until $Ra = 825$, where $t_0 \approx 27$. For values of the Rayleigh number beyond $Ra = 825$, the results obtained are erratic and we observed some negative values. As the Rayleigh number is increased, the characteristic time grows at a much faster rate than the initial transient time. The latter becomes less important to the fitting function and more prone to statistical errors originating from the finite size of the sample. Figure 11(b) shows the initial transient time normalized by the characteristic transient time as a function of $Ra$ and clearly shows that in the region where $t_0$ is poorly approximated, the values of $t_0$ returned by the least square fit remain small compared to the values of $\tau$: for $Ra > 825$, rare are the cases where the returned $t_0$ values are larger than 10% of $\tau$. 

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LOW-DIMENSIONAL TRANSITION MODEL

Close to the crisis bifurcation, the convection rolls do not vary much in size and occupy well-determined location in the physical domain. We can thus model the chaotic behavior in the vicinity of the crisis using a system of $N_r$ coupled oscillators, each corresponding to a roll described by two quantities: the convection energy, $c$, describing the energy contained in the $x$-component of the velocity and main component of any convective flow observed, and the tilt, $\tau$, describing the amount of tilt of a given roll or, similarly, the intensity of the $y$-component of the velocity. The quantity $c$ (resp. $\tau$) can be thought of as the analogue of $A_{\text{conv}}^2$ (resp. $\Phi_y$).

**One-roll system**

An essential feature of the doubly diffusive system is the presence of subcritical untilted states, as shown by the branches labelled C in the bifurcation diagram in figure 7. Such states can be modelled by a simple quadratic-cubic nonlinearity in a dynamical equation on $c$ and by simply damping out $\tau$. The uncoupled equations thus take the form:

\[
\begin{align*}
\partial_t c &= r c + c^2 - c^3, \\
\partial_t \tau &= -\gamma \tau,
\end{align*}
\]

(14)

(15)

where $r$ is the forcing parameter, akin to the Rayleigh number, and $\gamma \geq 0$ is the tilt decay rate. In the fluids problem, a destabilizing instability arises at a critical roll amplitude [37]. To model this, convection and tilt can be coupled in equation (15) such that when $c$ is larger than a threshold value (function of $\gamma$), the untilted states (with $\tau = 0$) become unstable. The coupled system then reads:

\[
\begin{align*}
\partial_t c &= r c + c^2 - c^3 - \beta c \tau^2, \\
\partial_t \tau &= -\gamma \tau + \frac{\beta}{2} c \tau,
\end{align*}
\]

(16)

(17)

where $\beta \geq 0$ is the rate of energy transfer between the tilt energy $\tau^2$ and the convection energy $c$. The effect of this coupling is to destabilize any solution $(c, \tau) = (c, 0)$ when $c > 2\gamma/\beta$ and, thus, to prevent the hysteresis by destabilizing the upper branch. The system presented above is similar to equations (16) and (17) of Bergeon and Knobloch [36],
with the exception of one term that is absent here and that was identified as having “[no] qualitative effect”.

System (16), (17) admits four steady states:

\[
\text{Cond} : (c, \tau) = (0, 0), \\
\text{Lower} : (c, \tau) = \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4r}, 0\right), \\
\text{Upper} : (c, \tau) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4r}, 0\right), \\
\text{Twist} : (c, \tau) = \left(\frac{2\gamma}{\beta}, \pm \sqrt{\frac{r}{\beta} + \frac{2\gamma}{\beta^2} - \frac{4\gamma^2}{\beta^3}}\right),
\]

where the subcritical state Lower bifurcates from the conductive solution, Cond, at \( r = 0 \) and extends down to \( r_s = -1/4 \), where it undergoes a stabilizing saddle-node bifurcation producing the upper branch labeled Upper. The state Twist bifurcates from either Lower or Upper depending on the value of the group \( 2\gamma/\beta \). In the fluids problem, this occurs when the roll energy is about 10% of the maximum roll energy within the snaking region, implying that \( 2\gamma/\beta \approx 0.1 \) and placing this bifurcation along the lower branch. The resulting bifurcation diagram is showed for \( \beta = 2 \) and \( \gamma = 0.1 \) in figure 12. The presence of the twist bifurcation alters the stability of the various steady states. The eigenmodes of the \( \tau = 0 \) states (Cond, Lower and Upper) are decoupled and correspond either to pure convection or tilt. The former is responsible for the primary bifurcation at \( r = 0 \) and destabilizes Lower for \(-1/4 < r < 0\), becoming stabilizing again at the saddle-node and creating a heteroclinic connection between Lower/Cond and Upper. Along the lower branch, the twist eigenmode becomes destabilizing at \( r_{\tau} = 4\gamma^2/\beta^2 - 2\gamma/\beta = -0.09 \), raising the dimension of the unstable manifold of both Lower for \( r < r_{\tau} \) and Upper. The bifurcation at \( r_{\tau} \) is the surrogate to the twist bifurcation in the doubly diffusive convection case and produces the solution Twist. The eigenmodes of Twist do not project simply on \( c \) or \( \tau \) and both eigenvalues collide to form a pair of complex conjugate eigenvalues at \( r_c = \delta^2(4\delta - 1)^2/(2\gamma) + 2\delta(2\delta - 1) = -0.082 \), where \( \delta = \gamma/\beta \).

For \( r > 0 \), there is no stable, \( c \geq 0 \) state and the dynamics takes the form of periodic orbits driven by the stable and unstable manifold of the three fixed points. Three examples of such stable periodic orbits are shown in figure 13. Starting from an initial condition near Cond, the orbit follows the unstable manifold of Cond, which is also the stable manifold of Upper, as the convection amplitude grows. The orbit is subsequently repelled by the unstable (twist)
FIG. 12. Bifurcation diagram for system (16), (17) with $\beta = 2$ and $\gamma = 0.1$. The steady solutions are represented via the energy $E = c + \tau^2$ versus the reduced Rayleigh number $r$. The stability of each solution is shown in the following way: $(\text{eig}_c, \text{eig}_\tau)$, where $\text{eig}_c$ (resp. $\text{eig}_\tau$) represents the sign of the growth rate of the convection (resp. twist) eigenmode. Although the eigenmodes of Twist are not so trivial, they have, for simplicity, been represented in the same fashion.

The manifold of Upper, which results in the increase of the twist energy, thereby suppressing convection and leading to the closure of the orbit as it approaches the stable manifold of Cond. This oscillatory behavior is supported by the oscillatory instability of Twist. The orbits shown in figure 13 approach Cond and Upper as $r$ is decreased, however, they are not formed at a global bifurcation at $r = 0$ but at a heteroclinic bifurcation just below $r = 0$. This was predicted by Bergeon & Knobloch in [36]: the case at hand is similar to their Fig. 15(a)[45]. To illustrate this, figure 14 shows the bifurcation diagram with the periodic orbit for $\gamma = 0.25$, where the global bifurcation is at an appreciable distance from $r = 0$. The periodic orbit collides with Lower and Upper at $r_{\text{het}} \approx -0.01$ in a heteroclinic bifurcation. As a result, there exists an $r$-interval, albeit small, in which the system is bistable.

$N_r$-roll system

There are various ways to couple rolls together and the point here is not to be exhaustive but rather to show that a simple coupling example can lead to chaos right above onset. We
consider a coupling term inspired by the way rolls break up after tilting. As a roll tilts, its convection amplitude decays, vanishing from the center towards the outside and forming, for a very short period of time preceding the eventual decay, two smaller tilted rolls (see Figures 5 and 6). Following the roll decay, two seeds of kinetic energy are left behind that may lead to the growth of two rolls instead of one at the newly vacated location. We take inspiration in this observation to propose the following coupled system:

\begin{align}
\partial_t c_i &= r c_i + c_i^2 - c_i^3 - \beta c_i \tau_i^2 + \eta(\tau_{i-1}^2 + \tau_{i+1}^2), \\
\partial_t \tau_i &= -\gamma \tau_i + \frac{\beta}{2} c_i \tau_i - \eta \tau_i,
\end{align}

where the subscript $i$ indicates the $i$-th roll, $i = 1, \ldots, N_r$, and $\eta \geq 0$. The coupling term redistributes the tilt energy of the $i$-th roll into the convection energy of the $i + 1$-th and $i - 1$-th rolls. More specifically, the term $-\eta \tau_i$ in equation (23) removes tilt energy at rate $-2\eta \tau_i^2$, half of which is reinjected in equation (22) for the $(i - 1)$-th roll, the other half being reinjected in the same equation but for the $(i + 1)$-th roll.

To model no-slip boundary conditions, we assume that the rolls at the extrema of the domain lose the same amount of tilt energy as central rolls but will only get fed convection
energy from one neighbor. We thus set \( \tau_0 = 0 \) and \( \tau_{N_r+1} = 0 \). To illustrate the complexity of the behavior obtained by the addition of this weak coupling, we set \( \beta = 2, \gamma = 0.1 \) and \( \eta = 0.01 \), and consider the Poincaré section corresponding to the conditions:

\[
\frac{\partial E}{\partial t} = \frac{\partial (\bar{c} + \bar{\tau}^2)}{\partial t} = 0, \quad \frac{\partial^2 E}{\partial t^2} > 0, \tag{24}
\]

where:

\[
\bar{c} = \sum_{i=1}^{N_r} c_i, \quad \bar{\tau}^2 = \sum_{i=1}^{N_r} \tau_i^2. \tag{25}
\]

Intersections of the trajectory with this Poincaré section correspond to local minima of the total energy of the system. Figure 15 shows the resulting diagram for system (22), (23) with 6 rolls. An initial run consisted in running simulations from \( r = -0.1 \) until \( r = 0.8 \) by steps of \( \Delta r = 5 \cdot 10^{-4} \). Each simulation was initialized by the last solution of the previous simulation except for the first simulation which was initialized using a random initial condition of small amplitude. The simulations are run for 15,000 time units and only the Poincaré intersections occurring during the last 5,000 time units are stored, thereby discarding transient dynamics. A second run was then carried out, starting from \( r = 0.8 \).
FIG. 15. Bifurcation diagram showing the value of the energy $E$ as a function of $r$ for a 6-roll system (22), (23) with $\beta = 2$, $\gamma = 0.1$ and $\eta = 0.01$. The points reported are Poincaré section intersects where the Poincaré plane is defined via the conditions $\partial_t E = 0$ and $\partial^2_t E > 0$. The diagram shows the density of Poincaré intersects: the more opaque, the denser.

and going down to $r = -0.1$ to identify how far the chaotic region would extend below $r = 0$. Although the initial run identified the presence of an immediate transition to temporal complexity at $r = 0$, the second run revealed that this chaotic behavior seems to persist down to $r \approx -0.044$, indicating a small region of coexistence between chaos and Cond. Figure 15 indicates a number of distinct regions exhibiting qualitatively different dynamics. Some of these regions are chaotic, as indicated by a diffuse set of Poincaré intersects, while others are dominated by periodic orbits and only feature Poincaré intersects at well-defined values of the energy $E$. To illustrate these different dynamics, several simulations are reported in figure 16. Panel (a) represents the dynamics at $r = 0$. The observed chaotic trajectory displays little structure besides the dominating anti-clockwise cycle resulting from the dynamics imposed by the uncoupled system (see figure 13). When $r < r_{\text{het}}$ in the uncoupled system, decay is unavoidable, but in the case of the coupled system (22), (23), the presence of a neighboring roll with non-zero tilt energy provides a source of convection energy capable of restarting a decaying roll. As $r$ is increased, the temporal dynamics becomes simpler and, by $r = 0.2$ (panel (c)), the trajectory follows two distinct stages. During the first stage, from the lowest energy point in the cycle to the point of maximum tilt energy $\tau^2$, the system follows straightforward dynamics: the convection energy $\tilde{c}$ first increases, leading to
FIG. 16. Phase space trajectory represented using the projections $\bar{c}$ and $\bar{\tau}^2$ for (a) $r = 0$, (b) $r = 0.16$, (c) $r = 0.2$ and (d) $r = 0.3$. Each of these trajectories represents the evolution of the system over 1,000 time units, after an initial transient of 199,000 time units has been discarded. Parameters are $N_r = 6$, $\beta = 2$, $\gamma = 0.1$ and $\eta = 0.01$.

The build up of tilt energy, which then suppresses convection. The second part of the cycle corresponds to the redistribution of twist energy into convection energy to neighboring rolls. As this happens, the trajectory spirals down to return to the point of minimal energy. This dynamical structure stems from bifurcations from the periodic orbit identified at $r = 0.16$ in panel (b). As the parameter is increased further, transitions to other attractors can be seen, such as the one occurring at $r \approx 0.28$ and leading to the dynamics shown in panel (d). This attractor is denser than the one shown in panel (c) and both coexist for a range
of parameter values. Lastly, this temporal complexity terminates at \( r \approx 0.75 \) where the last (Hopf) bifurcation leads to the return to a simple periodic orbit.

**DISCUSSION**

In this paper, we characterized the transition to chaos that takes place in doubly diffusive convection in a vertically extended domain. This transition occurs abruptly as the threshold for the linear instability of the conductive state is crossed. Linear instability from the conductive state first produces an array of counter-rotating convection rolls. Owing to the difference in diffusing time-scales between temperature and concentration, the rolls displaying downwelling flow at the hot wall are inhibited [39]. Past a certain amplitude, the resulting co-rotating rolls begin to tilt around the vertical axis and then decay. This secondary instability initiates chaotic dynamics characterized by the growth, tilt and decay of rolls at well-defined locations in the physical domain. Each roll appears to follow a cycle whereby they (i) grow from small amplitude residual energy, (ii) tilt due to the twist instability and then (iii) decay. Weak interactions between these rolls seem to be what is responsible for the temporal complexity observed by modifying the duration of each of the stages in the aforementioned cycle. This chaotic regime can be followed in the direction of decreasing Rayleigh number and it is found to disappear at a crisis bifurcation located in the vicinity of the primary bifurcation. The crisis bifurcation is abrupt, with a critical exponent of about 3, and yield very long chaotic transients in its vicinity. To illustrate the phenomenon, a low-dimensional model was constructed based on basic observations of the bifurcation structure of the doubly diffusive convection problem. The model takes advantage of the fact that the convection rolls always occupy the same position and do not drift, so that they can be represented by oscillators. Other choices regarding the oscillator coupling term than the one presented in this paper have been tested and also displayed transition to chaotic dynamics. This model exhibits a global bifurcation where time-dependent, long-lasting behavior arises at slightly subcritical values of the parameter and hinting at the possible existence of a small region of bistability between the conductive state and chaos. Owing to the large rate of increase of transient times in the vicinity of the crisis, assessing the existence and extent of this bistability region would represent a colossal numerical effort which was not attempted here.
This paper completes the study of the dynamics close to onset in doubly diffusive convection that was started in [37, 38]. While the above references elucidated subcritical pattern formation [37, 38], the transition to complex dynamics immediately at onset remained unclear. We have here shown this transition is the result of a crisis bifurcation, which we have characterized. Although crises are common in fluid dynamics [41, 46–48], to the author’s knowledge, it is the first time that it is observed to yield transition to chaos in the vicinity of the primary instability and with such a high critical exponent. The phenomenon identified in this paper only requires few conditions to take place: subcritical localized states subject to a secondary subcritical instability. These conditions can easily be found in large dimensional systems. For example, most fluid systems displaying steady localized pattern formation are both subcritical and of large extent. In such systems, if localized states are subject to a secondary instability, the scenario observed in this paper may be observable. As such, the results reported here may be of relevance to a large range of fluid systems.

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Here, $r$ (resp. $\gamma$) corresponds to their $\mu$ (resp. $\nu$), and where their $\gamma$ corresponds to the value of their $\mu$ at the global bifurcation, value that is always negative but very close to 0 for small $\nu$. 

