Supplementary Material for Convectons in a Rotating Fluid Layer

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Published in the Journal of Fluid Mechanics

Appendix B. Derivation of the fifth order Ginzburg-Landau equation

This file provides the pertinent details of the derivation of the nonlocal fifth order Ginzburg-Landau equation

\[ \mu A + A_{XX} + i (\gamma A_X + a_1 |A|^2 A_X + a_2 A^2 A_X^*) + b |A|^2 A - |A|^4 A = 0, \]  

(1)

and of the coefficients

\[ \mu = \mu_0 + \mu_1 \|A\|^2 + \mu_2 \|A\|^4 + \mu_3 \text{Im}[A A^*_X] + \mu_4 \|A\|^2, \]

\[ \gamma = \gamma_0 + \gamma_1 \|A\|^2, \quad b = b_0 + b_1 \|A\|^2 \]  

(2)


We start with the equations of motion (Veronis 1959)

\[ Ra \theta_x - T v_z + \nabla^4 \psi = \sigma^{-1} \left[ \nabla^2 \psi_t + J(\psi, \nabla^2 \psi) \right], \]

(3)

\[ \psi_x + \nabla^2 \theta = \theta_1 + J(\psi, \theta), \]

(4)

\[ T \psi_z + \nabla^2 v = \sigma^{-1} \left[ v_t + J(\psi, v) \right] \]

(5)

subject to the boundary conditions

\[ \psi = \psi_{zz} = \theta = v_z = 0 \text{ at } z \in \{0, 1\}. \]  

(6)

Equation (1) describes the above problem near the critical Rayleigh number \( Ra_c \) for the onset of convection and the critical Taylor number \( T_c \) determined by the degeneracy condition \( \xi^2 = 1/3 \) (Cox & Matthews 2001). Here \( \xi \equiv \frac{x^2}{\sqrt{4p^2 + 1}} \), \( p \equiv k^2 + \pi^2 \) and the wavenumber \( k \) represents the critical wavenumber corresponding to \( Ra_c \). Consequently, we write \( Ra = Ra_c + \epsilon^2 r_2 + \epsilon^4 r_4 \) and \( T = T_c + \epsilon^2 \delta \), where \( \epsilon \ll 1 \). We also introduce a slow spatial scale \( X \equiv \epsilon^2 x \) and write

\[ \psi \sim \sum_{n=1}^{\infty} \epsilon^n \psi_n(x, X, z), \quad \theta \sim \sum_{n=1}^{\infty} \epsilon^n \theta_n(x, X, z), \quad v \sim v_0(X) + \sum_{n=1}^{\infty} \epsilon^n v_n(x, X, z). \]  

(7)

The leading order term in the zonal velocity is now of order one instead of being of order \( \epsilon \) (Cox & Matthews 2001). To simplify the expressions that follow we use the notation

\[ \nabla^2 \equiv \partial_{xx} + \partial_{zz}, \quad J(u, w) \equiv u_x w_z - u_z w_x, \quad J(u, w) \equiv u_X w_z - u_z w_X, \]

\[ p_{nm} \equiv \left( \mu^2 k^2 + \mu^2 \pi^2 \right)^3 + Ra_c n^2 k^2 - T_c^2 m^2 \pi^2. \]

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Expansion (7) leads, order by order, to linear inhomogeneous problems of the form

\[ M \Psi_n \equiv \begin{pmatrix} \nabla^4 & Ra_c \partial_x & -T_c \partial_z \\ \partial_x & \nabla^2 & 0 \\ T_c \partial_z & 0 & \nabla^2 \end{pmatrix} \Psi_n = f_n, \]  

(8)

where \( \Psi_n \) represents \( (\psi_n, \theta_n, v_n)^T \) and \( f_n \) is a vector with components that are polynomials in \( \psi_1, \ldots, \psi_{n-1}, \theta_1, \ldots, \theta_{n-1}, v_0, \ldots, v_{n-1}, \) and their derivatives. Equation (8) can also be written as a single equation with respect to \( \psi_n \),

\[ M \psi_n \equiv \left( \nabla^6 - Ra_c \partial_{xx} + T_c^2 \partial_{zz} \right) \psi_n = \nabla^2 f_{n1} - Ra_c \partial_x f_{n2} + T_c \partial_z f_{n3}. \]  

(9)
We solve this equation for $\psi_n$ and determine the corresponding $\theta_n$ and $v_n$ from Eq. (8).

**First & second order**

At $O(\epsilon)\ f_1 = 0$. The resulting homogeneous problem has a solution of the form

$$
\psi_1 = \frac{a(X)}{2} e^{ikx} \sin(\pi z) + c.c.,
$$
$$
\theta_1 = \frac{ika(X)}{2p} e^{ikx} \sin(\pi z) + c.c.,
$$
$$
v_1 = \frac{\pi T_c a(X)}{2p} e^{ikx} \cos(\pi z) + c.c.,
$$

where $k$ and $\pi$ are, respectively, the wavenumbers in the $x$ and $z$ directions. The critical value $Ra_c$ and the critical wavenumber $k$ satisfy

$$
Ra_c = 3p^2, \quad p^2 \left(2k^2 - \pi^2\right) = T^2 \pi^2.
$$

(10)

Note that the linear operator $M$ in Eq. (9) is self-adjoint with kernel $e^{\pm ikx} \sin(\pi z)$. This fact simplifies the solvability condition applied at each subsequent order.

At $O(\epsilon^2)\ f_2$ is given by

$$
f_{21} = \sigma^{-1} J(\psi_1, \nabla^2 \psi_1),
$$
$$
f_{22} = -J(\theta_1, \psi_1),
$$
$$
f_{23} = -J(v_1, \psi_1).
$$

The solvability condition for $\psi_2$ is always satisfied. Thus $\psi_2$ may be set to zero and $\Psi_2$ is given by

$$
\psi_2 = 0,
$$
$$
\theta_2 = -\frac{k^2|a|^2}{8\pi p} \sin(2\pi z),
$$
$$
v_2 = v_{20}(X) + \frac{iT_c \pi^2 a^2}{16kp\sigma} e^{2ikx} + c.c.
$$

The homogeneous term $v_{20}(X)$ plays an important role in what follows.

**Third order**

At $O(\epsilon^3)\ f_3$ is given by

$$
f_{31} = -4 \nabla^2 \psi_{1,X} - Ra_c \theta_{1,X} + \delta v_{1,z} + \sigma^{-1} \left[J(\psi_2, \nabla^2 \psi_1) + J(\psi_1, \nabla^2 \psi_2)\right] - r_2 \theta_{1,X},
$$
$$
f_{32} = J(\psi_1, \theta_2) + J(\psi_2, \theta_1) - \psi_{1,X} - 2\theta_{1,X},
$$
$$
f_{33} = -\delta \psi_{1,z} - 2v_{1,X} + \sigma^{-1} \left[J(\psi_1, v_2) + J(\psi_2, \psi_1) - \psi_{1,z} v_{0,X}\right].
$$

The solvability condition at this order gives

$$
\left(\frac{k^2 r_2}{2} - T_c \pi^2 \delta\right) a - \frac{T_c \pi^2}{2\sigma} a v_{0,X} + \left(\frac{T^2 \pi^4}{16p^2} - \frac{3pk^4}{16}\right)|a|^2 a = 0.
$$

(11)

In traditional approaches which do not include spatial modulation the result (11) with $r_2 = \delta = 0$ reduces to $\xi^2 = 1$, i.e., the relation that determines the location of the codimension two point at which a subcritical periodic wavetrain becomes supercritical (Veronis 1959).
The solution $\Psi_3$ is

$$
\psi_3 = -\frac{T_2^2 \pi^4 a^3}{16p^4 \pi^3} e^{i k x} \sin(\pi z) - \frac{3p^4 |a|^2 a}{16p^3} e^{i k x} \sin(3\pi z) + c.c.,
$$

$$
\theta_3 = \frac{i k^3 (p + 3 \pi^2)}{16p^{13}(k^2 + 9\pi^2)} e^{i k x} \sin(3\pi z) - \frac{3i T_2^2 \pi^4 a^3}{16p^{13}\sigma^2(9k^2 + \pi^2)} e^{i k x} \sin(\pi z)
$$

$$
+ \frac{(\pi^2 - k^2)\sigma}{2p^3} e^{i k x} \sin(\pi z) - \frac{i k^3 |a|^2 a}{16p^2} e^{i k x} \sin(\pi z) + c.c.,
$$

$$
v_3 = \pi \left( v_{0,XX} + \delta \sigma \right) a + \frac{i T_2 \pi a X}{p^2} - \frac{T_2 \pi^3 |a|^2 a}{16p^2 \sigma^2} e^{i k x} \cos(\pi z)
$$

$$
- \frac{9p T_2 \pi^4 |a|^2 a}{16p^{13}(k^2 + 9\pi^2)} e^{i k x} \cos(3\pi z) - \frac{T_2 \pi^3 |a|^2 a}{16p^4 \pi^3} e^{i k x} \cos(\pi z) + c.c.
$$

**Fourth order**

At $O(\epsilon^4)$ $f_4$ is given by

$$
f_{41} = \sigma^{-1} \left[ 2J (\psi_1, \psi_1, \chi) + \tilde{J} (\psi_1, \nabla^2 \psi_1) + J (\psi_1, \nabla^2 \psi_3) + J (\psi_1, \nabla^2 \psi_1)
$$

$$
+ J(\psi_2, \nabla^2 \psi_2) - r_2 \theta_2, x - R_c \theta_2, x + 4v_2, x - 4\nabla^2 \psi_2, x, x,
$$

$$
f_{42} = -\psi_2, x - 2\theta_2, x - \tilde{J} (\theta_1, \psi_1) - J (\theta_3, \psi_1) - J (\theta, \psi_2) - J (\theta_1, \psi_3),
$$

$$
f_{43} = -\sigma^{-1} \left[ v_0, x \psi_2, x + \tilde{J} (v_1, \psi_1) + J (v_1, \psi_3) + J (v_2, \psi_2) + J (v_3, \psi_1)
$$

$$
- 2v_2, x - v_0, x - \delta \psi_2, x.
$$

The solvability condition for $v_4$ is always satisfied at this order. However, the Laplace operator appearing in the equation for $v_4$ has a nonempty kernel spanned by multiples of $v_4 = 1$, a fact that is responsible for the presence of a second solvability condition, viz.,

$$
v_{0,XX} + \frac{T_2 \pi^2 (|a|^2)}{4p \sigma} = 0.
$$

From Eqs. (11) and (12) we obtain the condition $\xi^2 = 1/3$ that defines the critical Taylor number $T_c$:

$$
T_c = T_c^{\text{mod}} \equiv \frac{\sigma \pi^2 (2 \pm \sqrt{1 - \sigma^2})}{(1 \pm \sqrt{1 - \sigma^2})^2}.
$$

Note that two such critical values of $T$ are present.

Now that all solvability conditions have been imposed we can proceed to solve the $O(\epsilon^4)$ problem. Here we only list the terms which enter into the computation of the
solvability conditions arising at $O(\varepsilon^5)$ and $O(\varepsilon^6)$. These are

$$\psi_4 = \frac{k^2 \pi (2T_x^2 \pi^2 + 4p^2 \pi^2 + 3\sigma p^3)(|a|^2)_{XX} \sin(2\pi z)}{2p^2p_{02}\sigma} \sin(2\pi z) + ..., $$

$$\theta_4 = \frac{k^3(p_{13} - 3p^2k^2)(k^2 + 5\pi^2)|a|^4}{32\pi p^2p_{13}(k^2 + 9\pi^2)} \sin(2\pi z) + \left\{ \frac{ik\pi^2a}{8p^2} \sin(2\pi z) + c.c. \right\} + ..., $$

$$v_4 = \frac{k^4(4T_x^2\pi^2 + 8p^2\pi^2 + 6\sigma p^3 + p_{02})(|a|^2)_{XX} \cos(2\pi z)}{8pp_{02}\pi^2}
+ \left\{ \frac{-ik\pi^2(T_x^2\pi^2 + p_{13})(5k^2 + \pi^2)|a|^2a^2}{64pp_{31}\sigma^2(9k^2 + \pi^2)} e^{2ikx} + \frac{i\pi^2(v_0, X + \delta\sigma)a^2}{16kp\sigma^2} e^{2ikx}
- \frac{(3k^2 + \pi^2)|a|^2}{16p} e^{2ikx} + c.c. \right\} + .... $$

**Fifth & sixth order**

At $O(\varepsilon^5)$ and $O(\varepsilon^6)$, $f_5$ and $f_{63}$ are given by

$$f_{51} = \sigma^{-1}\left[ \sum_{n=1}^{2} \left( 2J(\psi_4, \psi_{3-n}, Xx) + J(\psi_4, \nabla^2\psi_{3-n}) \right) \right] - 4\psi_{1,XX}xx
- 2\nabla^2\psi_{1,XX} - 4\nabla^2\psi_{3,XX} - Ra_1\theta_{3,XX} - r_4\theta_{4,XX} + \delta\psi_{3,z} - r_2\theta_{4,XX} - r_2\theta_{3,XX},$$

$$f_{52} = -\psi_{3,XX} + 2\theta_{3,XX} - \sum_{n=1}^{4} J(\theta_n, \psi_{5-n}) - \sum_{n=1}^{2} J(\theta_n, \psi_{3-n}),$$

$$f_{53} = -\sigma^{-1}\left[ \sum_{n=1}^{4} J(\psi_5, \psi_{5-n}) + \sum_{n=1}^{2} J(\psi_n, \psi_{3-n}) + v_0, X\psi_{3,z} \right] - v_{1,XX} + 2v_{3,XX} - \delta\psi_{3,z},$$

$$f_{63} = -\sigma^{-1}\left[ \sum_{n=1}^{3} J(\psi_n, \psi_{4-n}) + \sum_{n=1}^{5} J(\psi_n, \psi_{6-n}) + v_0, X\psi_{4,z} \right] - v_{2,XX} + 2v_{4,XX} - \delta\psi_{4,z}.$$ 

The solvability condition for $\psi_5$ yields

$$\left( \bar{\mu}_0 + \bar{\mu}_1 v_{0,XX} + i\bar{\mu}_2 v_{0,XX} \right) a + \bar{\delta}a_{XX} + i \left[ (\bar{\gamma}_0 + \bar{\gamma}_1 v_{0,XX}) a + \bar{\alpha}_{10}|a|^2a + \bar{\alpha}_{20}|a|^2 \delta a \right]$$
\[+ \left( \bar{b}_0 + \bar{b}_1 v_{0,XX} \right) |a|^2a - 2\bar{c}_0|a|^2a - \frac{p\bar{k}^2}{2} \alpha_{20,XX} = 0 \quad (14) \]

with coefficients given by

$$\bar{\mu}_0 = \frac{k^2\pi - \delta^2\pi^2}{2}, \quad \bar{\mu}_1 = -\frac{\delta\pi^2}{2\sigma}, \quad \bar{\delta} = 6k^2p, \quad \bar{\gamma}_0 = -\frac{k\pi^2(r_2 + 2T_c\delta)}{p},$$

$$\bar{\mu}_2 = \bar{\gamma}_1 = -k^3, \quad \bar{b}_0 = \frac{2p\delta^2\pi^2 - k^2r^2\sigma}{16p\sigma} k^2, \quad \bar{b}_1 = \frac{k^2\pi^2}{16\sigma^2},$$

$$\bar{\alpha}_{10} = \frac{k^3p(4k^2 + 2\pi^2 + 3\sigma)(2k^4 - 9\pi^4 + 7k^2\pi^2 + 6k^2\pi^4\sigma)}{8pp_{02}\sigma^2} + \frac{k^3(3\pi^2 + k^2)}{32} + \frac{k^7}{16\pi^2},$$

$$\bar{\alpha}_{20} = \frac{k^3p(4k^2 + 2\pi^2 + 3\sigma)(2k^4 - 9\pi^4 + 7k^2\pi^2 + 6k^2\pi^4\sigma)}{8pp_{02}\sigma^2} + \frac{k^3(3\pi^2 + k^2)}{32} + \frac{k^7}{16\pi^2},$$

$$\bar{c}_0 = \frac{p^2k^6(19k^2 + 2\pi^2)}{128p_{31}(9k^2 + \pi^2)} + \frac{k^4\pi^2(5k^2 + 2\pi^2)}{64\pi^2(9k^2 + \pi^2)} + \frac{9k^8p^2}{128p_{13}} - \frac{3k^6(2k^2 + 5\pi^2)(p_{13} - 3k^2\pi^2)}{64p_{13}(9k^2 + \pi^2)}. $$
The solvability condition for \( v_6 \) yields

\[
v_{20,XX} + \left( \frac{k^3}{2p} \text{Im}[aa^*_X] + \frac{\pi^2|a|^2 v_{9,X}}{4p\sigma^2} + \frac{\delta\pi^2|a|^2}{4p\sigma} - \frac{k^2\pi^2|a|^4}{32p\sigma^2} \right)_X = 0.
\]

Equation (1) follows from (14) on integrating conditions (12) and (15) with respect to \( X \).

REFERENCES
